Relating Income to Consumption
Part 1

Extract from "Earnings, Consumption and Lifecycle Choices"
by Costas Meghir and Luigi Pistaferri.
Handbook of Labor Economics, Vol. 4b, Ch. 9. (2011).

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AEA Continuing Education Program
ASSA Course: Microeconomics of Life Course Inequality
San Francisco, CA, January 5-7, 2016

Introduction

- The objective of this chapter is to discuss recent developments in the literature that studies how the dynamics of earnings and wages affect consumption choices over the life cycle.
- Labor economists and macroeconomists are the main contributors to this area of research.
- A theme of interest for both labor economics and macroeconomics is to understand how much risk households face, to what extent risk affects basic household choices such as consumption, labor supply and human capital investments, and what types of risks matter for explaining behavior.
- These are questions that have a long history in economics.

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- A fruitful distinction is between *ex-ante* and *ex-post* household responses to risk.
- *Ex-ante* responses answer the question:
  "What do people do in the anticipation of shocks to their economic resources?".
- *Ex-post* responses answer the question:
  "What do people do when they are actually hit by shocks to their economic resources?".

The Impact of Income Changes on Consumption: Theory

The Life Cycle-Permanent Income Hypothesis

- To see how the degree of persistence of income shocks and the nature of income changes affects consumption, consider a simple example in which income is the only source of uncertainty of the model.
- Preferences are quadratic, consumers discount the future at rate $\frac{1-\beta}{\beta}$ and save on a single risk-free asset with deterministic real return r, $\beta(1+r)=1$ (this precludes saving due to returns outweighing impatience), the horizon is finite (the consumer dies with certainty at age A and has no bequest motive for saving), and credit markets are perfect.
- Quadratic preferences are traditional, but restrictive.

The Life Cycle-Permanent Income Hypothesis

- $c_{i,a,t}$ is consumption at age a and time t; $y_{i,a,t}$ is income at age a and time t
- The change in household consumption can be written as

$$\Delta c_{i,a,t} = \pi_a \sum_{j=0}^{A} \frac{E\left(y_{i,a+j,t+j}|\Omega_{i,a,t}\right) - E\left(y_{i,a+j,t+j}|\Omega_{i,a-1,t-1}\right)}{(1+r)^j} \tag{1}$$

a indexes age and t time,

- $\pi_a = \frac{r}{1+r}\left[1 - \frac{1}{(1+r)^{A-a+1}}\right]^{-1}$ is an "annuity" parameter that increases with age and $\Omega_{i,a,t}$ is the consumer's information set at age a.
- This expression is rich enough to identify three key issues regarding the response of consumption to changes in the economic resources of the household.

- First, consumption responds to news in the income process, but not to expected changes.
- The second key issue emerging from equation (1) is that the life cycle horizon also plays an important role (the term π_a).

- The last key feature of equation (1) is the persistence of innovations.
- More persistent innovations have a larger impact than short-lived innovations.
- To give a more formal characterization of the importance of persistence, suppose that income follows an ARMA(1,1) process:

$$y_{i,a,t} = \rho y_{i,a-1,t-1} + \varepsilon_{i,a,t} + \theta \varepsilon_{i,a-1,t-1} \tag{2}$$

- In this case, substituting (2) in (1), the consumption response is given by

$$\Delta c_{i,a,t} = \left(\frac{r}{1+r}\right)\left[1-\frac{1}{(1+r)^{A-a+1}}\right]^{-1}$$
$$\left[1+\frac{\rho+\theta}{1+r-\rho}\left(1-\left(\frac{\rho}{1+r}\right)^{A-a}\right)\right]\varepsilon_{i,a,t}$$
$$=\kappa\left(r,\rho,\theta,A-a\right)\varepsilon_{i,a,t}$$

- Table 1 shows the value of the marginal propensity to consume κ for various combinations of ρ, θ, and $A - a$ (setting $r = 0.02$).

Table 1: The response of consumption to income shocks under quadratic preferences

ρ	θ	$A-a$	κ
1	-0.2	40	0.81
1	0	10	1
0.99	-0.2	40	0.68
0.95	-0.2	40	0.39
0.8	-0.2	40	0.13
0.95	-0.2	30	0.45
0.95	-0.2	20	0.53
0.95	-0.2	10	0.65
0.95	-0.1	40	0.44
0.95	-0.01	40	0.48
1	0	∞	1
0	-0.2	40	0.03

- A number of facts emerge.
- If the income shock represents an innovation to a random walk process ($\rho = 1, \theta = 0$), consumption responds one-to-one to it regardless of the horizon (the response is attenuated only if shocks end after some period, say $L < A$).
- A decrease in the persistence of the shock lowers the value of κ. When $\rho = 0.8$ (and $\theta = -0.2$) for example, the value of κ is a modest 0.13.

- A decrease in the persistence of the MA component acts in the same direction (but the magnitude of the response is much attenuated).
- In this case as well, the presence of liquidity constraints may invalidate the sharp prediction of the model.
- For example, more and less persistent shocks may have a similar effect on consumption.
- When the consumer is hit by a short-lived negative shock, she can smooth the consumption response over the entire horizon by borrowing today (and repaying in the future when income reverts to the mean).
- If borrowing is precluded, a short-lived or long-lived shock have similar impacts on consumption.

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- The income process (2) considered above is restrictive, because there is a single error component which follows an ARMA(1,1) process.
- A very popular characterization in calibrated macroeconomic models is to assume that income is the sum of a random walk process and a transitory i.i.d. component:

$$y_{i,a,t} = p_{i,a,t} + \varepsilon_{i,a,t} \quad (3)$$
$$p_{i,a,t} = p_{i,a-1,t-1} + \zeta_{i,a,t} \quad (4)$$

- The appeal of this income process is that it is close to the process used in a Friedman's permanent income hypothesis.

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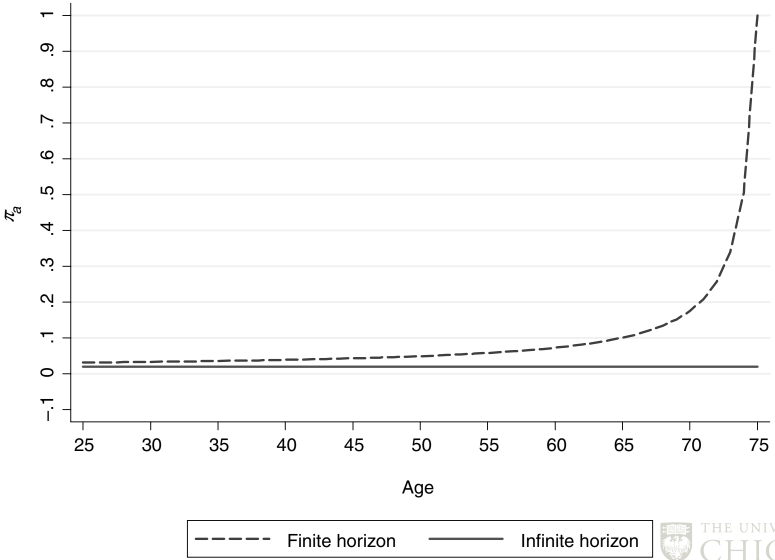
- In this case, the response of consumption to the two types of shocks is:

$$\Delta c_{i,a,t} = \pi_a \varepsilon_{i,a,t} + \zeta_{i,a,t} \quad (5)$$

- Consumption responds one-to-one to permanent shocks but the response of consumption to a transitory shock depends on the time horizon.
- For young consumers (with a long time horizon), the response should be small.
- The response should increase as consumers age.

- Figure 1 plots the value of the response for a consumer who lives until age 75.

Figure 1: The response of consumption to a transitory income shock



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- Clearly, it is only in the last 10 years of life or so that there is a substantial response of consumption to a transitory shock.
- The graph also plots for the purpose of comparison the expected response in the infinite horizon case.
- An interesting implication of this graph is that a transitory unanticipated stabilization policy is likely to affect substantially only the behavior of older consumers (unless liquidity constraints are important—which may well be the case for younger consumers).

- Note finally that if the permanent component were literally permanent ($p_{i,a,t} = p_i$), it would affect the level of consumption but not its change.
- In the classical version of the LC-PIH the *size* of income changes does not matter.
- One reason why the size of income changes may matter is because of adjustment costs: Consumers tend to smooth consumption and follow the theory when expected income changes are large, but are less likely to do so when the changes are small and the cost of adjusting consumption are not trivial.

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- Suppose for example that consumers who want to adjust their consumption upwards in response to an expected income increase need to face the cost of negotiating a loan with a bank.
- This "magnitude hypothesis" has been formally tested by Scholnick (2010), who use a large data set provided by a Canadian bank that includes information on both credit cards spending as well as mortgage payment records.
- As in Stephens (2008) he argues that the final mortgage payment represent an expected shock to disposable income (that is, income net of pre-committed debt service payments).

- Outside the quadratic preference world, uncertainty about future income realizations will also impact consumption.

Beyond the PIH

- The model with quadratic preferences gives very sharp predictions regarding the impact on consumption of various types of income shocks.
- For example, there is the sharp prediction that permanent shocks are entirely consumed (an MPC of 1).
- Unfortunately, quadratic preferences have well known undesirable features, such as increasing risk aversion with wealth and lack of a precautionary motive for saving.
- Do the prediction of this model survive under more realistic assumptions about preferences?
- The answer is: only qualitatively.

- The problem with more realistic preferences, such as CRRA, is that they deliver no closed form solution for consumption — that is, there is no analytical expression for the "consumption function" and hence the value of the propensity to consume in response to risk (income shocks) is not easily derivable.
- This is also the reason why the literature moved on to estimating Euler equations after Hall (1978).
- The advantage of the Euler equation approach is that one can be silent about the sources of uncertainty faced by the consumer (including crucially the stochastic structure of the income process).

- However, in the Euler equation approach only a limited set of parameters (preference parameters such as the elasticity of intertemporal substitution or the intertemporal discount rate) can be estimated.
- Some dissatisfaction in the literature regarding the evidence coming from Euler equation estimates (see Browning and Lusardi, 1996; Attanasio and Weber, 2010).

- Recently there has been an attempt to go back to the concept of a "consumption function".
- Two approaches have been followed.
- First, the Euler equation that describe the expected dynamics of the growth in the marginal utility can be approximated to describe the dynamics of consumption growth.
- Blundell, Pistaferri and Preston (2008), extending Blundell and Preston (1998) (see also Blundell and Stoker, 1994), derive an approximation of the mapping between the expectation error of the Euler equation and the income shock.
- Second, there is work by Carroll (2001) and Kaplan and Violante (2010) discuss numerical simulations in the buffer-stock and Bewley model, respectively.
- We discuss the results of these two approaches in turn.

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Approximation of the Euler equation

- Blundell, Pistaferri and Preston (2008) consider the consumption problem faced by household i of age a in period t.
- Assuming that preferences are of the CRRA form, the objective is to choose a path for consumption C so as to:

$$\max_{C} E_a \sum_{j=0}^{A-a} \beta^j \frac{C_{i,a+j,t+j}^{1-\gamma} - 1}{1-\gamma} e^{Z'_{i,a+j,t+j}\vartheta_{a+j}}. \tag{6}$$

where $Z_{i,a+j,t+j}$ incorporates taste shifters (such as age, household composition, etc.), and we denote with $E_a(.) = E(.|\Omega_{i,a,t})$.

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- Maximization of (6) is subject to the budget constraint which in the self-insurance model assumes individuals have access to a risk free bond with real return r

$$A_{ia+j+1} = (1+r)\left(A_{i,a+j,t+j} + Y_{i,a+j,t+j} - C_{i,a+j,t+j}\right) \quad (7)$$
$$A_{i,A} = 0 \quad (8)$$

with $A_{i,a,t}$ given.

- Blundell, Pistaferri and Preston (2008) set the retirement age after which labor income falls to zero at L, assumed known and certain, and the end of the life-cycle at age A.

- They assume that there is no uncertainty about the date of death.
- With budget constraint (7), optimal consumption choices can be described by the Euler equation (assuming for simplicity that there is no preference heterogeneity, or $\vartheta_a = 0$):

$$C_{i,a-1,t-1}^{-\gamma} = \beta (1+r) E_{a-1} C_{i,a,t}^{-\gamma}. \tag{9}$$

- As it is, equation (9) is not useful for empirical purposes.

- Blundell, Pistaferri and Preston (2008) show that the Euler equation can be approximated as follows:

$$\Delta \log C_{i,a,t} \simeq \eta_{i,a,t} + f^{C}_{i,a,t}$$

where $\eta_{i,a,t}$ is a consumption shock with $E_{a-1}(\eta_{i,a,t}) = 0$, $f^{c}_{i,a,t}$ captures any slope in the consumption path due to interest rates, impatience or precautionary savings and the error in the approximation is $O(E_a \eta^2_{i,a,t})$.

- Suppose that any idiosyncratic component to this gradient to the consumption path can be adequately picked up by a vector of deterministic characteristics $\Gamma^{c}_{i,a,t}$ and a stochastic individual element $\xi_{i,a,t}$

$$\Delta \log C_{i,a,t} - \Gamma^{c}_{i,a,t} = \Delta c_{i,a,t} \simeq \eta_{i,a,t} + \xi_{i,a,t}.$$

- Assume log income is

$$\log Y_{i,a,t} = p_{i,a,t} + \varepsilon_{i,a,t} \quad (10)$$
$$p_{i,a,t} = \Gamma^{y}_{i,a,t} + p_{i,a-1,t-1} + \zeta_{i,a,t} \quad (11)$$

where $\Gamma^{y}_{i,a,t}$ represent observable characteristic influencing the growth of income.

- Income growth can be written as:

$$\Delta \log Y_{i,a,t} - \Gamma^{y}_{i,a,t} = \Delta y_{i,a,t} = \zeta_{i,a,t} + \Delta\varepsilon_{i,a,t}.$$

- The (ex-post) intertemporal budget constraint is

$$\sum_{j=0}^{A-a} \frac{C_{i,a+j,t+j}}{(1+r)^j} = \sum_{j=0}^{L-a} \frac{Y_{i,a+j,t+j}}{(1+r)^j} + A_{i,a,t}$$

where A is the age of death and L is the retirement age.

- Applying the approximation above and taking differences in expectations gives

$$\eta_{i,a,t} \simeq \Xi_{i,a,t} \left[\zeta_{i,a,t} + \pi_a \varepsilon_{i,a,t} \right]$$

where π_a is an annuitization factor defined below in technical notes, $\Xi_{i,a,t} = \frac{\sum_{j=0}^{A-a} \frac{Y_{i,a+j,t+j}}{(1+r)^j}}{\sum_{j=0}^{A-a} \frac{Y_{i,a+j,t+j}}{(1+r)^j} + A_{i,a,t}}$ is the share of future labor income in current human and financial wealth, and the error of the approximation is $O([\zeta_{i,a,t} + \pi_a \varepsilon_{i,a,t}]^2 + E_{a-1} [\zeta_{i,a,t} + \pi_a \varepsilon_{i,a,t}]^2)$.

Link to Technical Appendix

- Then

$$\Delta \log C_{i,a,t} \simeq \xi_{i,a,t} + \Xi_{i,a,t}\zeta_{i,a,t} + \pi_a \Xi_{i,a,t}\varepsilon_{i,a,t} \qquad (12)$$

with a similar order of approximation error.

- The random term $\xi_{i,a,t}$ can be interpreted as the innovation to higher moments of the income process.

- The interpretation of the impact of income shocks on consumption growth in the PIH model with CRRA preferences is straightforward.
- For individuals a long time from the end of their life with the value of current financial assets small relative to remaining future labor income, $\Xi_{i,a,t} \simeq 1$, and permanent shocks pass through more or less completely into consumption whereas transitory shocks are (almost) completely insured against through saving.
- Precautionary saving can provide effective self-insurance against permanent shocks only if the stock of assets built up is large relative to future labor income, which is to say $\Xi_{i,a,t}$ is appreciably smaller than unity, in which case there will also be some smoothing of permanent shocks through self insurance.

- The most important feature of the approximation approach is to show that the effect of an income shock on consumption depends not only on the persistence of the shock and the planning horizon (as in the LC-PIH case with quadratic preferences), but also on preference parameters.
- *Ceteris paribus*, the consumption of more prudent households will respond less to income shocks.
- The reason is that they can use their accumulated stock of precautionary wealth to smooth the impact of the shocks (for which they were saving precautiously against in the first place).
- Simulation results (below) confirm this basic intuition.

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Kaplan and Violante

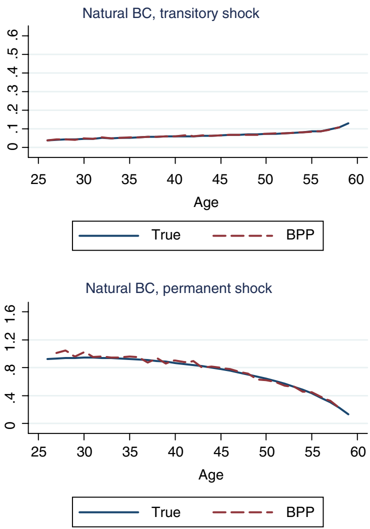
- Kaplan and Violante (2010) investigate the amount of consumption insurance present in a life-cycle version of the standard incomplete markets model with heterogenous agents (e.g., Rios-Rull, 1995; Huggett, 1996).
- Kaplan and Violante's setup differs from that in Blundell, Pistaferri and Preston by adding the uncertainty component μ_a to life expectancy, and by omitting the taste shifters from the utility function.
- μ_a is the probability of dying at age a.
- It is set to 0 for all $a < L$ (the known retirement age) and it is greater than 0 for $L \leq a \leq A$.

- The KV model also differs from BPP by specifying a realistic social security system.
- Two baseline setups are investigated - a natural borrowing constraint setup (henceforth NBC), in which consumers are only constrained by their budget constraint, and a zero borrowing constraint setup (henceforth ZBC), in which consumers have to maintain non-negative assets at all ages.
- The income process is similar to BPP.
- Part of KV's analysis is designed to check whether the amount of insurance predicted by the Bewley model can be consistently estimated using the identification strategy proposed by BPP and whether BPP's estimates using PSID and CEX data conform to values obtained from calibrating their theoretical model.

- Kaplan and Violante (2010) model is calibrated to match the US data.
- Survival rates are obtained from the NCHS, the intertemporal discount rate is calibrated to match a wealth-income ratio of 2.5, the permanent shock parameters (σ_ζ^2 and the variance of the initial draw of the process) are calibrated to match PSID data and the variance of the transitory shock (σ_ε^2) is set to the 1990-1992 BPP point estimate (0.05).
- The KV model is solved numerically.
- This allows for the calculation of both the "true" and the BPP estimators of the "partial insurance parameters" (the response of consumption to permanent and transitory income shocks).

- Figure 2 is reproduced from Kaplan and Violante (2010).

Figure 2: Age profile of MPC coefficients for transitory and permanent income shocks. (Source: Kaplan and Violante (2010)).



- First, in the NBC environment the MPC with respect to transitory shocks is fairly low throughout the life cycle, and similarly to what is shown in Figure 1, increases over the life cycle due to reduced planning horizon effect. The life cycle average MPC is 0.06.
- Second, there is considerable insurance also against the permanent shock, which increases over the life cycle due to the ability to use the accumulated wealth to smooth these shocks. The life cycle average MPC is 0.77, well below the MPC of 1 predicted by the infinite horizon PIH model.

- Third, the ZBC environment affects only the ability to insure transitory shocks (which depend on having access to loans), but not the ability to insure permanent shocks (which depend on having access to a storage technology, and hence it is not affected by credit restrictions).
- Fourth, the performance of the BPP estimators is remarkably good. Only in the case of the ZBC environment and the permanent shock does the BPP estimator display an upward bias, and even in that case only very early in the life cycle.
- According to KV the source of the bias is the failure of the orthogonality condition used by BPP for agents close to the borrowing constraint.
- It is worth noting that the ZBC environment is somewhat extreme as it assumes no unsecured borrowing.

- Finally, KV compare the average MPCs obtained in their model (0.06 and 0.77) with the actual estimates obtained by BPP using actual data.
- As we shall see, BPP find an estimate of the MPC with respect to permanent shocks of 0.64 (s.e. 0.09) and an estimate of the MPC with respect to transitory shocks of 0.05 (s.e. 0.04).
- Clearly, the "theoretical" MPCs found by KV lie well in the confidence interval of BPP's estimates.

- One thing that seems not to be borne out in the data is that theoretically the degree of smoothing of permanent shocks should be strictly increasing and convex with age, while BPP report increasing amount of insurance with age as a non-significant finding.
- As discussed by Kaplan and Violante (2010), the theoretical pattern of the smoothing coefficients is the result of two forces: a wealth composition effect and a horizon effect.
- The increase in wealth over the life cycle due to precautionary and retirement motives means that agents are better insured against shocks.
- As the horizon shortens, the effect of permanent shock resembles increasingly that of a transitory shock.
- Given that the response of consumption to shocks of various nature is so different (and so relevant for policy in theory and practice), it is natural to turn to studies that analyze the nature and persistence of the income process.

Technial Appendix

Technical Discussion Drawn from Blundell, Low, and Preston (2008)

Approximating the Euler Equation

- The household plan at age t is to maximize the expected remaining lifetime utility:

$$E_t \sum_{\tau=0}^{T-t} \frac{U(c_{i,t+\tau})}{(1+\delta)^{\tau}}$$

- We begin by calculating the error in approximating the Euler equation.

$$E_t U'(c_{it+1}) = U'(c_{it}) \left(\frac{1+\delta}{1+r} \right) = U'\left(c_{it} e^{\kappa_{it+1}}\right) \qquad (13)$$

for some κ_{it+1}

- By exact Taylor expansion of period $t+1$ marginal utility in $\ln c_{it+1}$ and around $\ln c_{it} + \kappa_{it+1}$, there exists a $\tilde{c}$ between $c_{it}e^{\kappa_{it+1}}$ and c_{it+1} such that

$$U'(c_{it+1}) = U'\left(c_{it}e^{\kappa_{it+1}}\right)\Bigg[1 + \frac{1}{\kappa\left(c_{it}e^{\kappa_{it+1}}\right)}\left[\Delta \ln c_{it+1} - \kappa_{it+1}\right] + \frac{1}{2}\beta\left(\tilde{c}, c_{it}e^{\kappa_{it+1}}\right)\left[\Delta \ln c_{it+1} - \kappa_{it+1}\right]^2\Bigg] \quad (14)$$

where $\kappa(c) \equiv U'(c)/cU''(c) < 0$ and $\beta(\tilde{c}, c) \equiv [\tilde{c}^2 U'''(\tilde{c}) + \tilde{c}U''(\tilde{c})]/U'(c)$.

- Taking expectations

$$E_t U'(c_{it+1}) = U'\left(c_{it} e^{\kappa_{it+1}}\right)\left[1 + \frac{1}{\kappa\left(c_{it} e^{\kappa_{it+1}}\right)} E_t\left[\Delta \ln c_{it+1} - \kappa_{it+1}\right] + \frac{1}{2} E_t\left\{\beta\left(\tilde{c}, c_{it} e^{\kappa_{it+1}}\right)\left[\Delta \ln c_{it+1} - \kappa_{it+1}\right]^2\right\}\right] \quad (15)$$

- Substituting for $E_t U'(c_{it+1})$ from (13)

$$\frac{1}{\kappa(c_{it} e^{\kappa_{it+1}})} E_t \left[\Delta \ln c_{it+1} - \kappa_{it+1}\right] + \frac{1}{2} E_t \left\{\beta(\tilde{c}, c_{it} e^{\kappa_{it+1}}) \left[\Delta \ln c_{it+1} - \kappa_{it+1}\right]^2\right\} = 0 \quad (16)$$

and thus

$$\Delta \ln C_{it+1} = \kappa_{it+1} - \frac{\kappa(c_{it} e^{\kappa_{it+1}})}{2} E_t \left\{\beta\left(\tilde{c}, c_i t e^{\kappa_{it+1}}\right) \left[\Delta \ln c_{it} + 1 e^{\kappa_{it+1}}\right]^2\right\} + \varepsilon_{it+1} \quad (17)$$

where the consumption innovation ε_{it+1} satisfies $E_t \varepsilon_{it+1} = 0$. As $E_t \varepsilon^2_{it+1} \to 0$, $\beta(\tilde{c}, c_{it} e^{\kappa_{it+1}})$ tends to a constant and therefore by Slutsky's theorem

$$\Delta \ln c_{it+1} = \varepsilon_{it+1} + \kappa_{it+1} + \mathcal{O}\left(E_t \left|\varepsilon_{it+1}\right|^2\right). \quad (18)$$

- If preferences are CRRA then κ_{it+1} does not depend on $c_i t$ and is common to all households, say κ_{t+1}.
- The log of consumption therefore follows a martingale process with common drift

$$\Delta \ln c_{it+1} = \varepsilon_{it+1} + \kappa_{t+1} + \mathcal{O}\left(E_t \left|\varepsilon_{it+1}\right|^2\right). \quad (19)$$

Approximating the Lifetime Budget Constraint

- The second step in the approximation is relating income risk to consumption variability.
- In order to make this link between the consumption innovation ε_{it+1} and the permanent and transitory shocks to the income process, we loglinearise the intertemporal budget constraint using a general Taylor series approximation (extending the idea in Campbell 1993).

- Define a function F: $\mathbb{R}^{N+1} \to \mathbb{R}$ by $F(\boldsymbol{\xi}) = \ln \sum_{j=0}^{N} \exp \xi_j$.
- By exact Taylor expansion around an arbitrary point $\boldsymbol{\xi}^0 \in \mathbb{R}^{N+1}$

$$F(\boldsymbol{\xi}) = \ln \sum_{j=0}^{N} \exp \xi_j^0 + \sum_{j=0}^{N} \frac{\exp \xi_j^0}{\sum_{k=0}^{0} \exp \xi_k^0} (\xi_j - \xi_j^0) + \frac{1}{2} \sum_{j=0}^{N} \sum_{k=0}^{N} \frac{\partial^2 F(\tilde{\boldsymbol{\xi}})}{\partial \xi_j \partial \xi_k} \left(\xi_j - \xi_j^0\right) \left(\xi_k - \xi_k^0\right) \qquad (20)$$

where $\tilde{\boldsymbol{\xi}}$ lies between $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^0$ and is used to make the expansion exact.

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- The coefficients in the remainder term are given by

$$\frac{\partial^2 F(\tilde{\boldsymbol{\xi}})}{\partial \xi_j \partial \xi_k} = \frac{\exp \tilde{\xi}_j}{\sum_k \exp \tilde{\xi}_k} \left(\delta_{jk} - \frac{\exp \tilde{\xi}_j}{\sum_k \exp \tilde{\xi}_k} \right), \qquad (21)$$

where δ_{jk} denotes the Kronecker delta.

- These coefficients are bounded because $0 < \exp \tilde{\xi}_j / \sum_k \exp \tilde{\xi}_k < 1$.

- Hence, taking expectations of (21) subject to information set $\mathcal{I}$

$$E_{\mathcal{I}}\left[F(\boldsymbol{\xi})\right] = \ln \sum_{j=0}^{N} \exp \xi_j^0 + \sum_{j=0}^{N} \frac{\exp \xi_j^0}{\sum_{k=0}^{N} \exp \xi_k^0} (E_{\mathcal{I}}\xi_j - \xi_j^0) + \frac{1}{2} \sum_{j=0}^{N} \sum_{k=0}^{N} E_{\mathcal{I}} \left(\frac{\partial^2 F(\tilde{\boldsymbol{\xi}})}{\partial \xi_j \partial \xi_k} (\xi_j = \xi_j^0)(\xi_k - \xi_k^0) \right). \tag{22}$$

- We apply this expansion firstly to the expected present value of consumption, $\sum_{j=0}^{T-t} c_{it+j}(1+r)^{-j}$.
- Let $N = T - t$ and let

$$\begin{aligned} \xi_j &= \ln c_{it+j} - j \ln(1+r) \\ \xi_j^0 &= E_{t-1} \ln c_{it+j} - j \ln(1+r), \quad i = 0, \dots, T-t. \end{aligned} \tag{23}$$

- Then, substituting equation (23) into equation (22) and noting only the order of magnitude for the remainder term,

$$E_{\mathcal{I}}\left[\ln\sum_{j+0}^{T-t}\frac{c_{it+j}}{(1+r)^{j}}\right]=\ln\sum_{j+0}^{T-t}\exp\left[E_{t-1}\ln c_{it+j}-j\ln(1+r)\right]$$
$$+\sum_{j=0}^{T-t}\theta_{it+j}\left[E_{\mathcal{I}}\ln c_{it+j}-E_{t-1}\ln c_{it+j}\right]$$
$$+\mathcal{O}(E_{\mathcal{I}}\|\varepsilon_{it}^{T}\|^{2}) \qquad (24)$$

where

$$\theta_{it+j}=\frac{\exp\xi_j^0}{\sum_{k=0}^{N}\exp\xi_k^0}=\frac{\exp\left[E_{t-1}\ln c_{it+j}-j\ln(1+r)\right]}{\sum_{k=0}^{T-t}\exp\left[E_{t-1}\ln c_{it+k}-kl(1-r)\right]}$$

and ε_{it}^{T} denotes the vector of future consumption innovations $(\varepsilon_{it},\varepsilon_{it+1},\ldots,\varepsilon_{iT})'$.

- The term θ_{it+j} can be seen as an annuitisation factor for consumption.
- We now apply the expansion (22) to the expected present vale of resources, $\sum_{j=0}^{R-t-1}(1+r)^{-j}y_{it+j} + A_{iT+1}(1+r)^{-(T-t)}$.
- Let $N = R - T$ and let

$$\begin{aligned}
\xi_j &= \ln y_{it+j} - j\ln(1-r) \\
\xi_j^0 &= E_{t-1}\ln y_{it+j} - j\ln(1+r) \quad j = 0, \ldots, R-t-1 \\
\xi_N &= \ln\left[A_{it} - A_{iT+1}(1+r)^{-(T-t)}\right] \\
\xi_N^0 &= E_{t-1}\ln[A_{it} - A_{iT-1}(a+r)^{-(T-t)}].
\end{aligned} \tag{25}$$

- Then, substituting equation (25) into equation (22), and again noting only the order of magnitude for the remainder term,

$$
\begin{aligned}
&E_{\mathcal{I}} \ln \left(\sum_{j=0}^{R-r-1} \frac{y_{it+j}}{(1+r)^j} + A_{it} - \frac{A_{iT+1}}{(1+r)^{T-t}} \right) = \\
&\ln \left[\sum_{j=0}^{R-t-1} \exp \left[E_{t-1} \ln y_{it+j} - j \ln(1+r) \right] + \exp E_{t-1} \ln \left[A_{it} - \frac{A_{iT+1}}{(1+r)^{T-t}} \right] \right] \\
&+ \pi_{it} \sum_{j=0}^{R-t-1} \alpha_{t+j} \left[E_{\mathcal{I}} \ln y_{it+j} - E_{t-1} \ln y_{it+j} \right] \\
&+ (1-\pi_{it}) \left[E_{\mathcal{I}} \ln \left[A_{it} - \frac{A_{iT+1}}{(1+r)^{T-1}} \right] - E_{t-1} \ln \left[A_{it} - \frac{A_{iT+1}}{(a+r)^{T-t}} \right] \right] \\
&+ \mathcal{O} \left(E_{t-1} \| (\boldsymbol{\nu}_{it}^{R-1}) \|^2 \right) \qquad (26)
\end{aligned}
$$

where

$$\pi_{t+j} =$$

$$= \frac{\exp[E_{t-1} \ln y_{it+j} - j \ln(1+r)]}{\sum_{k=0}^{R-t-1} \exp[E_{t-1} \ln y_{it+k} - k \ln(1+r)]}$$

$$= \frac{\exp\left[\sum_{k=0}^{j} (\eta_{t+k} + E_{t-1} \bar{u}_{t+k}) + E_{t-1} \bar{u}_{t+j} - j \ln(1+r)\right]}{\sum_{k=0}^{R-t-1} \exp\left[\sum_{t=0}^{k} (\eta_{t+l} + E_{t-1} \bar{u}_{t+l}) + E_{t-1} \bar{u}_{t+k} - k \ln(1+r)\right]}$$

(This is the same as $\pi_{i,a,t}$ in Blundell et al., 2008.)

. . . can be seen as an annuitisation factor for income (common within a cohort because of the assumption of common income trends) and

$$\Xi_{i,a,t} = 1 - \frac{\exp \xi_N^0}{\sum_{k=0}^{N} \exp \xi_k^0}$$

$$= \frac{\sum_{j=0}^{R-t-1} \exp[E_{t-1} \ln y_{t+j} - j \ln(1+r)]}{\sum_{j=0}^{R-t-1} \exp[E_{t-1} \ln y_{it+j} - j \ln(1+r)] + \exp E_{t-1} \ln[A_{it} - A_{iT+1}/(1+R)^{T-t}]}$$

is (roughly) the share of expected future labor income in current human and financial wealth (net of terminal assets) and $\boldsymbol{\nu}_{it}^{R-1}$ denotes the vector of future income shocks $(\nu'_{it}, \nu'_{it+1}, \ldots, \nu'_{iR-1})'$.

This corresponds to the $\Xi_{i,a,t}$.

- We are able to equate the subjects of equations (24) and (26) because the realised budget must balance and $\sum_{j=0}^{R-t} \frac{c_{it+j}}{(1+r)^j}$ and $\sum_{j=0}^{R-t-1} \frac{y_{it+j}}{(1+r)^j} + A_{it} - \frac{A_i T+1}{(1+r)^{T-t}}$ therefore have the same distribution.

- We use (24) and (26), taking differences between expectations at the start of the period, before the shocks are realised, and at the end of the period, after the shocks are realised.
- This gives

$$\begin{aligned} \varepsilon_{it} &+ \mathcal{O}(E_t \parallel \boldsymbol{\varepsilon}_{it}^T \parallel^2 + E_{t-1} \parallel \boldsymbol{\varepsilon}_{it}^T \parallel^2) \\ &= \pi_{it}(v_{it} + \alpha_t u_{it}) + \pi_{it}\Omega_t \\ &\quad + \mathcal{O}(E_t \parallel \boldsymbol{\nu}_{it}^{R-1} \parallel^2 + E_{t-1} \parallel \boldsymbol{\nu}_{it}^{R-1} \parallel^2), \end{aligned}$$

where the left hand side is the innovation to the expected present value of consumption and the right hand side is the innovation to the expected present value of income and

$$\Omega_t = \sum_{j=0}^{R-t-1} \pi_{t+j} \sum_{k=0}^{j} (E_t - E_{t-1})\omega_{t+k}$$

captures the revision to expectations of current and future common shocks.

- Squaring the two sides, taking expectations and inspecting terms reveals that the terms which are $\mathcal{O}(E_t \parallel \varepsilon_{it}^T \parallel^2 + E_{t-1} \parallel \varepsilon_{it}^T \parallel^2)$ are $\mathcal{O}(E_t \parallel \boldsymbol{\nu}_{it}^{R-1} \parallel^2 + E_{t-1} \parallel \boldsymbol{\nu}_{it}^{R-1} \parallel^2)$.
- Furthermore, since, for all $j \geq 0, \parallel \nu_{it+j} \parallel^2 = \mathcal{O}_p(E_t \parallel \nu_{it+j} \parallel^2)$ by Chebyshev's inequality, $E_t \parallel \boldsymbol{\nu}_{it}^{R-1} \parallel^2 = \mathcal{O}_p(E_{t-1} \parallel \boldsymbol{\nu}_{it}^{R-1} \parallel^2)$.
- Thus

$$\varepsilon_{it} = \Xi_{i,a,t}(v_{it} + \pi_{i,t} u_{it}) + \Xi_{i,a,t}\Omega_t + \mathcal{O}_p(E_{t-1} \parallel \boldsymbol{\nu}_{it}^{R-1} \parallel^2)$$

and therefore

$$\Delta \ln c_{it} = \kappa_t + \Xi_{i,a,t}(v_{it} + \alpha_t u_{it}) + \Xi_{i,a,t}\Omega_t + \mathcal{O}_p(E_{t-1} \parallel \boldsymbol{\nu}_{it}^{R-1} \parallel^2).$$

End of Digression.

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