Static Labor Force Participation

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Static Model of Married Women’s Labor Force Participation

Let

\[ a = \text{the age of the wife} \]
\[ C_a = \text{household consumption} \]
\[ P_a = 1 \text{ if wife works, } = 0 \text{ if wife does not work} \]
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\[ U_a = U_a(C_a, P_a) \]

\( U(0, 0) = 0, \ U_C > 0, \ U_{CC} < 0, \ U(C, 1) < U(C, 0), \ \frac{\partial U}{\partial C} \bigg|_{P=0} \leq \frac{\partial U}{\partial C} \bigg|_{P=1} \)

Marginal utility from consumption may increase/decrease with labor participation.
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Budget Constraint:

\[ C_a = y_a + w_a P_a, \]
Alternative-Specific Utilities:

\[ U^1 = U(y + w, 1) \]
\[ U^0 = U(y, 0) \]
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Decision Rule (maximize utility subject to the budget constraint):

\[ P = 1 \quad \text{iff} \quad U(y + w, 1) \geq U(y, 0) \]
\[ = 0 \quad \text{otherwise} \]
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Participation Function:

\[ P = P(y, w) \]
Are there testable implications of the model?
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Assume a cross section of married couples with different ages of the wife for whom we have the following information:

\[ P_{ai}, y_{ai}, w_{ai}, \quad i = 1, \ldots, I \]

Assume wage offer data are available even both for those women who work and for those who do not. Then,
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Assume wage offer data are available even both for those women who work and for those who do not. Then,

\[(i) \text{ If } y_{aj} = y_{ak} \text{ and } w_{aj} = w_{ak} \text{ then } P_{aj} = P_{ak}\]
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Assume a cross section of $I$ married couples with different ages of the wife for whom we have the following information:

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\begin{align*}
(i) \quad & \text{If } y_{aj} = y_{ak} \text{ and } w_{aj} = w_{ak} \text{ then } P_{aj} = P_{ak} \\
(ii) \quad & \text{If } y_{aj} = y_{ak}, \ w_{aj} > w_{ak} \text{ then } P_{aj} \geq P_{ak}
\end{align*}

These may not be true in the data.
Why might the theory be rejected?
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(i) Theory is correct, but $w$, $y$, and/or $P$ are incorrectly measured.
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(ii) $P$ depends on more than just $w$ and $y$, e.g., preferences for the wife’s leisure differ among households with the same $y$ and $w$. 
Why might the theory be rejected?

(i) Theory is correct, but \( w, y, \) and/or \( P \) are incorrectly measured.

(ii) \( P \) depends on more than just \( w \) and \( y \), e.g., preferences for the wife’s leisure differ among households with the same \( y \) and \( w \).

(iii) The theory is wrong in a fundamental way – the assumption of static optimization is wrong, households do not optimize.
(i) Any theory can be reconciled with data if we allow for unrestricted measurement error.

Need external validation of the true values of $y$, $w$ and $P$. 
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Need external validation of the true values of $y$, $w$ and $P$.

(ii) Allow for preferences to depend on other household characteristics that might affect the value of the wife’s leisure, e.g., the number of young children in the household. The rejection criterion would condition on these characteristics.
But, there are likely be *unmeasured* characteristics that are related to household preferences.

How can we explicitly account for them?
One way is to explicitly account for the possibility that couples differ in the value they place on the wife’s leisure in ways that the researcher cannot measure.
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Denoting the unobserved preference for leisure by $\epsilon$, write

$$U = U(C, P; \epsilon (1 - P))$$

where

$$U_{\epsilon|P=0} > 0$$
Then, the participation decision is:

\[ P = 1 \text{ iff } U(y + w, 1) \geq U(y, 0; \epsilon) \]

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or,

\[ P = 1 \text{ iff } \epsilon \leq \epsilon^*(y, w) \]

\[ P = 0 \text{ otherwise} \]

where \( \epsilon \leq \epsilon^*(y, w) \) is the threshold value of \( \epsilon \) at which the couple is just indifferent between \( P=1 \) and \( P=0 \).
Properties of the cut-off value, $\varepsilon^*(y, w)$:

\[
\frac{\partial \varepsilon^*(y, w)}{\partial w} > 0 \quad \frac{\partial \varepsilon^*(y, w)}{\partial y} \leq 0
\]

As $y$ goes up, the value of wife staying at home may go up/down.
Let $\epsilon \sim F_{\epsilon|y,w}$. Then, participation is probabilistic from our perspective, though not from the household’s.

\[
\Pr(P = 1|y, w) = \int_{-\infty}^{\epsilon^*(y,w)} dF_{\epsilon|y,w}
\]

\[
= F_{\epsilon|y,w}(\epsilon^*(y,w)) = G(y, w)
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$$= F_{\epsilon|y,w}(\epsilon^*(y,w)) = G(y, w)$$

Note that $G(y, w)$ is a composite of $F$ and $U$, both of which depend on $y$ and $w$. Thus, comparative static effects of $w$ and $y$ on the participation probability confound changes in the utility of participation and non-participation and changes in the distribution of unmeasured preferences.
Are there testable implications?
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If we assume that $\epsilon$ is distributed independently of $w$ and $y$, that is, $F_{\epsilon|y,w} = F_\epsilon$,

$$\frac{\partial \Pr(P = 1|y,w)}{\partial w} = G_w(y,w) = f_\epsilon(\epsilon^*(y,w)) \frac{\partial \epsilon^*}{\partial w} > 0$$

$$\frac{\partial \Pr(P = 1|y,w)}{\partial y} = G_y(y,w) = f_\epsilon(\epsilon^*(y,w)) \frac{\partial \epsilon^*}{\partial y} \leq 0$$
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The model thus has the testable implications that the *probability* of working is increasing in the wage level. Testing that proposition requires estimation.
Estimation:

The participation function is thus $P(y, w; \epsilon)$. It is determined by the primitives $U$ and $F$ and the assumption of maximization. The participation probability function, $G(y, w)$, is itself not a primitive.
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Questions:

1. What can we learn about the features of $U$, $F$ and $G$ given cross sectional data on $P$, $y$ and $w$?
Estimation:

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Questions:

1. What can we learn about the features of $U$, $F$ and $G$ given cross sectional data on $P$, $y$ and $w$?

2. How does what we learn depend on *a priori* assumptions?
Approaches to Estimation – A General Taxonomy
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Parametric vs. Non-Parametric:

Parametric approaches impose either a functional form assumption on $U$ or $G$, and/or a distributional assumption on $F$.

Non-Parametric approaches usually restrict the functions to be a member of a broad class, such as the class of continuous and differentiable functions.
Approaches to Estimation – A General Taxonomy

Parametric vs. Non-Parametric: *Parametric* approaches impose either a functional form assumption on $U$ or $G$, and/or a distributional assumption on $F$. *Non-Parametric* approaches usually restrict the functions to be a member of a broad class, such as the class of continuous and differentiable functions.

Structural vs. Non-Structural:

*Structural* approaches recover the primitive functions, $U$ and $F$.

*Non-structural* approaches recover $G$. 
The following table describes the approaches that we will consider for the labor force participation model.

<table>
<thead>
<tr>
<th></th>
<th>nonparametric (NP)</th>
<th>parametric (P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>nonstructural (NS)</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>structural (S)</td>
<td>X</td>
<td>✓</td>
</tr>
</tbody>
</table>

Note: See Matzkin (1992, Ecma) for the NP-S case.
NP-NS Approach:
NP-NS Approach

Note that

\[
E(P|y, w) = \Pr(P = 1|y, w) \cdot 1 + [1 - \Pr(P = 1|y, w)] \cdot 0
\]

\[
= \Pr(P = 1|y, w)
\]
NP-NS Approach

Note that

\[
E(P|y,w) = \Pr(P = 1|y,w) \cdot 1 + [1 - \Pr(P = 1|y,w)] \cdot 0
\]

\[= \Pr(P = 1|y,w)\]

Thus,

\[P = E(P|y,w) + u = G(y,w) + u\]

where \(E(u|y,w) = 0\) by construction \((u=P-E(u|y,w))\). The \(G\) function can be estimated by non-parametric regression.
NP-NS Approach

Note that

\[ E(P|y,w) = \Pr(P = 1|y,w) \cdot 1 + [1 - \Pr(P = 1|y,w)] \cdot 0 \]

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Thus,

\[ P = E(P|y,w) + u = G(y,w) + u \]

where \( E(u|y,w) = 0 \) by construction. The G function can be estimated by non-parametric regression.

The model can be tested by checking whether

\[ G_w(w,y) > 0 \text{ for all } w, y. \]
P-NS Approach:
Choose a functional form for $G$. Examples are:

1. Linear

$$G(w, y) = \alpha_0 + \alpha_1 y + \alpha_2 w$$

which yields the linear probability model

$$P_i = \alpha_0 + \alpha_1 y_i + \alpha_2 w_i + u_i.$$
P-NS Approach

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which yields the linear probability model

$$P_i = \alpha_0 + \alpha_1 y_i + \alpha_2 w_i + u_i$$

A test of the model is whether $\alpha_2 > 0$. 
P-NS Approach

2. a. Probit

\[ G = \Phi(\alpha_0 + \alpha_1 y_i + \alpha_2 w_i) \]

where \( \Phi \) is the standard normal cumulative.
P-NS Approach

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   b. Logit
   
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   where \( \Lambda \) is the logistic function \( \exp(x)/(1 + \exp(x)) \).
P-NS Approach

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where \( \Lambda \) is the logistic function \( \exp(x)/(1 + \exp(x)) \).

Denoting the parameters of the G function by \( \Theta^G \), the likelihood function is
\[ L(\Theta^G; \text{data}) = \prod_{i=1}^{I} G(y_i, w_i)^{P_i} [1 - G(y_i, w_i)]^{1-P_i} \]
P-S Approach:
P-S Approach

Denoting the parameters of the utility function by $\Theta^U$ and those of the distribution function by $\Theta^F$,

$$L(\Theta^U, \Theta^F; \text{data}) = \prod_{i=1}^{I} F_\epsilon(\epsilon^*(y, w))^{P_i} [1 - F_\epsilon(\epsilon^*(y, w))]^{1-P_i}$$
P-S Approach

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\]

We will consider three alternatives.
1. Assume a functional form for $U$, which will imply a form for $\epsilon^* (y, w)$, but leave the distribution of $F$ free.
P-S Approach

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2. Assume a distribution for $F$ and leave $U$ free.
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2. Assume a distribution for $F$ and leave $U$ free.

3. Assume a functional form for both $U$ and $F$. 
P-S Approach

Cases 1 and 3: As an illustration, assume the utility function to be

\[ U(C, P; \epsilon) = C + \alpha_1 (1 - P) + \alpha_2 (1 - P)C + \epsilon (1 - P) \]
P-S Approach

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Alternative-specific utilities:

\[ U^1 = y + w \]

\[ U^0 = y + \alpha_1 + \alpha_2 y + \epsilon \]
P-S Approach

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Alternative-specific utilities:

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Threshold value for \( \epsilon, \epsilon^* (y, w) : \)

\[ \epsilon^* (y, w) = w - \alpha_2 y - \alpha_1 \]
Case 1:

\[ L(\text{data}) = \prod_{i=1}^{I} G(y_i, w_i)^{P_i} [1 - G(y_i, w_i)]^{1-P_i} \]

where

\[ G(y_i, w_i) = F_\epsilon(w - a_2 y_2 - a_1) \]
P-S Approach

Case 3: Assume further that the preference for leisure is normally distributed \((\epsilon \sim N(0, \sigma_\epsilon^2))\). Then,

\[
G(y, w) = \Phi\left(\frac{w - \alpha_2 y - \alpha_1}{\sigma_\epsilon}\right)
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where \(\Phi\) is the standard cumulative normal.
P-S Approach

Case 3: Assume further that the preference for leisure is normally distributed \( (\epsilon \sim N(0, \sigma^2_\epsilon)) \). Then,

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What is the test of the model?
P-S Approach

Case 3: Assume further that the preference for leisure is normally distributed ($\epsilon \sim N(0, \sigma_\epsilon^2)$). Then,

$$G(y, w) = \Phi\left(\frac{w - \alpha_2 y - \alpha_1}{\sigma_\epsilon}\right)$$

where $\Phi$ is the standard cumulative normal.

A test of the model comes down to the condition that

$$\frac{\partial \text{Pr}(P = 1|w, y)}{\partial w} = \sigma_\epsilon^{-1} \phi\left(\frac{w - \alpha_2 y - \alpha_1}{\sigma_\epsilon}\right) > 0 \text{ for all } y, w.$$
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\]

This condition is equivalent to a test that the coefficient on the wage, \(\sigma_\epsilon^{-1}\), is positive.
P-S Approach

Case 2: Specify a distribution for $\epsilon$ and non-parametrically estimate $\epsilon^*(y, w)$.

For example, if $\epsilon \sim N(0, \sigma^2_\epsilon)$, it is possible to identify $\epsilon^*(y, w)$ up to scale, that is, up to a normalization of $\sigma_\epsilon$. 
P-S Approach

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For example, if $\varepsilon \sim N(0, \sigma^2_{\varepsilon})$, it is possible to identify $\varepsilon^*(y, w)$ up to scale, that is, up to a normalization of $\sigma_{\varepsilon}$.

But, non-parametric estimation $\varepsilon^*(y, w)$ does not non-parametrically identify $U$. 
How should we choose the estimation method, that is, one set of assumptions vs. another?
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What is the goal of estimation?

1. provide an understanding the determinants of the labor supply decision.
2. perform policy evaluation
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1. provide an understanding the determinants of the labor supply decision.
2. perform policy evaluation

Conditional on achieving the goal of estimation, the fewer extra-theoretic assumptions the better. Are there tradeoffs between achieving a goal and reliance on extra-theoretic assumptions?
Ex-Ante Policy Evaluation

predicting the impact of policies that are outside the historical experience.
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Is it possible within a non-parametric framework?
Ex-Ante Policy Evaluation

predicting the impact of policies that are outside the historical experience.

Is it possible within a non-parametric framework?

Consider, for example, the imposition of a wage tax, where none had existed before and suppose we non-parametrically estimate $G(y, w)$
A proportional wage tax changes the budget constraint:

\[ C = y + (1 - \tau)wP \]

\[ = y + w'P \]

*What is the participation probability under the new tax regime?*
A proportional wage tax changes the budget constraint:

\[ C = y + (1 - \tau)wp \]
\[ = y + w'P \]

The participation probability with the tax is \( G(y, w') \).
A proportional wage tax changes the budget constraint:

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\[ = y + w'P. \]

The participation probability with the tax is \( G(y, w') \).

Letting \( h(w, y) \) be the sample density of \( w, y \), the effect of the tax on the participation rate in the sample is

\[ \int_{S_w} \int_{S_y} [G(w', y) - G(w, y)]h(w, y)dwdy \]

where \( S_w \) and \( S_y \) denote the support of \( w \) and \( y \).
NP-NS: Ex-Ante Policy Evaluation

population tax effect = sample tax effect
  only if
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NP-NS: Ex-Ante Policy Evaluation

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If, for example, the supports are not the same, then it will be necessary to extrapolate outside the range of support observed in the sample using a P-NS or P-S approach.
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If, for example, the supports are not the same, then it will be necessary to extrapolate outside the range of support observed in the sample using a P-NS or P-S approach.

Even if the supports are the same, there will always be values of $w$ for which there are no observation with a wage of $w'$. For example, the minimum wage in the population has no analog in $w'$ (The new minimum wage is lower than the minimum wage observable in the old regime).
P-S: Ex-Ante Policy Evaluation

P-S: The effect of the wage tax in the P-S approach is

$$\Phi\left(\frac{(1 - \tau)w - \alpha_2 y - \alpha_1}{\sigma_\epsilon}\right) - \Phi\left(\frac{w - \alpha_2 y - \alpha_1}{\sigma_\epsilon}\right)$$

Identification of $\sigma_\epsilon$ is critical for performing the policy evaluation.
Ex-Ante Policy Evaluation

If one has sufficient data, non-parametric estimation of the policy effect is preferred.

The P-S approach provides a more precise estimate, but may not fit the observations well.
Model vs Data: An Example

To fix ideas, we begin with the static labor force participation model.

\[ U_a(C_a, P_a; x_a) = C_a + \alpha_1 (1 - P_a) + \alpha_2 C_a (1 - P_a) + \alpha_3 (1 - P_a) n_a \]
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C: consumption,
P: participation
n: number of young children
Let c be the cost of childcare if the woman works.
Let y be husband’s income.
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s: years of schooling,
a: age
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\xi_a \sim N(0, \sigma^2_{\xi})
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Household’s state space: \( \Omega_a = \{ y_a, n_a, s, a, \xi_a \} \)
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Household’s state space:

\[ \Omega_a = \{ y_a, n_a, s, a, \xi_a \} \]

State space observable to the researcher:

\[ \Omega_a^- = \{ y_a, n_a, s, a \} \]
Alternative-Specific Utilities:

\[ U^1 = y_a - cn_a + \exp(\gamma_0 + \gamma_1 s + \gamma_2 a - \gamma_3 a^2 + \xi_a), \]

\[ U^0 = y_a + \alpha_1 + \alpha_2 y_a + \alpha_3 n_a. \]
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Decision Rule:

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P_a = 1 \text{ if } \xi_a \geq \log(\alpha_1 + \alpha_2 y_a + (\alpha_3 + c)n_a) - \gamma_0 - \gamma_1 s - \gamma_2 a + \gamma_3 a^2 = \xi^*_a(\Omega_a)
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= 0, \text{ otherwise.}
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The woman works if, given her age, education, husband’s income and number of children, her \( \xi_a \) exceeds the critical value \( \xi^*(\Omega^-) \).
Data: Cross-section of $i = 1, \ldots, I$ married women

$P_{ia}, w_{ia}P_{ia}, y_{ia}, n_{ia}, s_i$ and $a_i$
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Model parameters:

$\Theta = \{\alpha_1, \alpha_2, \alpha_3, c, \gamma_0, \gamma_1, \gamma_2, \gamma_3, \sigma_\xi^2\}$
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Likelihood function:

$$L(\Theta; \text{data}) = \prod_{i=1}^{I}[\Pr(P_{ia} = 1, w_{ia}|\Omega_{ia}^{-})]^{P_{ia}}[\Pr(P_{ia} = 0)|\Omega_{ia}^{-})]^{1-P_{ia}}$$
Pr(P_{ia} = 1, w_{ia} | Ω_{ia}^{-}) = Pr(P_{ia} = 1 | w_{ia}, Ω_{ia}^{-}) Pr(w_{ia} | Ω_{ia}^{-}).
Rewrite

\[
Pr(P_{ia} = 1, w_{ia} | \Omega_{ia}^-) = Pr(P_{ia} = 1 | w_{ia}, \Omega_{ia}^-) \cdot Pr(w_{ia} | \Omega_{ia}^-).
\]

The first term, the probability of working conditional on the wage, is

\[
Pr(w_{ia} > \alpha_1 + \alpha_2 y_{ia} + (\alpha_3 + c) n_{ia} | w_{ia}, \Omega_{ia}^-)
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What values can this probability take?
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What values can this probability take?

Only 0 or 1. Why?
Either the condition inside the probability statement holds or it does not hold.

What happens to the likelihood?
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The likelihood will not be degenerate (zero) only if there exists a set of parameters such that this condition is not violated (even for a single observation).
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Thus, the parameters must satisfy the set of restrictions that

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Thus, the parameters must satisfy the set of restrictions that

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In particular, it must be satisfied for the person with the lowest observed wage. Clearly that observation will have an extreme effect on the parameter estimates.
How can we avoid this?
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1. Add an additional error term to the decision model. For example, we can assume that $\alpha_1$ differs across people, i.e., replace $\alpha_1$ with $\alpha_{1i} = \alpha_1 + \epsilon_i$. 
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1. Add an additional error term to the decision model.

   For example, we can assume that $\alpha_1$ differs across people, i.e., replace $\alpha_1$ with $\alpha_{1i} = \alpha_1 + \epsilon_i$

What complication does this raise?

We lose the analytical solution for the cutoff value.

$$\xi_a - \log(\alpha_1 + \epsilon_i + \alpha_2 y_a + (\alpha_3 + c) n_a) - \gamma_0 - \gamma_1 s - \gamma_2 a + \gamma_3 a^2 \leq 0$$
2. Assume that the wage data are measured with error, for example, that the observed (reported) wage measures the true wage with a proportionate error,

\[ \log w_{ia}^o = \log w_{ia} + \eta_{ia}, \]

where

\[ \eta_{ia} \sim N(0, \sigma_\eta^2) \text{ and } E(\xi_{ia} \eta_{ia}) = 0. \]
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What advantage does this have over the previous method?

The decision rule is unchanged, so that the cutoff value has an analytical form. The parameter restriction that we had before only holds with respect to the true wage, not with respect to reported wages.
How does the likelihood function change?

\[
\Pr(P_{ia} = 1, w_{ia} | \Omega_{ia}^{-}) = (w_{ia}^{o})^{-1} \Pr(\xi_{ia} \geq \xi_{ia}^{*}(\Omega_{ia}^{-}), u_{ia} = \log w_{ia}^{o} - (\gamma_{0} + \gamma_{1}s_{i} + \gamma_{2}a_{i} - \gamma_{3}a_{i}^{2}))
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How does the likelihood function change?

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\]

where \( u_{ia} = \xi_{ia} + \eta_{ia} \).

Given normality, this becomes

\[
\Pr(P_{ia} = 1, w^o_{ia} | \Omega^-_{ia}) = (1 - \Phi(\frac{\xi^*_{ia}(\Omega^-_{ia}) - \rho \frac{\sigma_\xi}{\sigma_u} u_{ia}}{\sigma_\xi \sqrt{1 - \rho^2}})) \frac{1}{\sigma_u} \phi\left(\frac{u_{ia}}{\sigma_{u_{ia}}}\right),
\]
where $\sigma_u = \sqrt{\sigma_\xi^2 + \sigma_u^2}$, $\rho = \sigma_\xi / \sigma_u$ and $1 - \rho^2$ is the fraction of the variance in the wage due to measurement error.
\[ \Pr(P_{ia} = 1, w_{ia}^o | \Omega_{ia}^-) = (1 - \Phi(\frac{\xi_{ia}^* (\Omega_{ia}^-) - \rho \frac{\sigma_{\xi}}{\sigma_u} u_{ia}}{\sigma_{\xi} \sqrt{1 - \rho^2}})) \frac{1}{\sigma_u} \phi\left(\frac{u_{ia}}{\sigma_{u_{ia}}}\right), \]

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The component of the likelihood function for non-workers is

\[ \Pr(P_a = 0) = \Phi(\xi_{ia}^* / \sigma_{\xi}) \]